A new numerical scheme for Einstein equations with discrete variational derivative method

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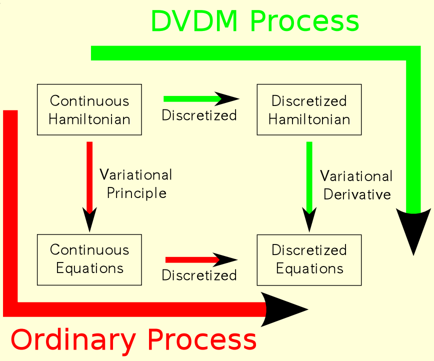
Motivations

- What is the best way to make a discretized equations for Numerical Relativity (NR)?
- In NR, the Crank-Nicolson (CN) scheme and Runge-Kutta scheme (RK) are often used.
- However, these schemes were not proposed for NR.
- For accuracy simulations, we need to use a numerical scheme that is built for NR.

Method

- The Discrete Variational Derivative Method (DVDM) is one of the numerical scheme.
- The DVDM was proposed and extended by Furihata, Mori and Matsuo (D. Furihata and T. Matsuo, *Discrete Variational Derivative Method*, (CRC press, 2010)).
- The DVDM is considered as a discrete version of the variational principle.
- To make a discretized equations using the DVDM scheme, the Lagrangian or the Hamiltonian is necessary.
- With the DVDM scheme, we can make a discretized equations with preserving constraints and diffusion characters in the continuous system.



The diagram of making the discretized equations from the continuous equations. In general, the equations are derived from the Hamiltonian by the variational principle, and the discretized equations using numerical schemes such as the CN scheme or the RK scheme (red line process). On the other hand, by the DVDM scheme, we first make a discrete Hamiltonian, and derive the discretized equations (green line process).

Application to Einstein Equations

We apply the DVDM scheme to the canonical formalism of the Einstein equations. A discrete Hamiltonian density of the Einstein equations can be written as

$$\mathcal{H}^{\mathrm{GR}(n)}_{(k)} = -\alpha^{(n)}_{(k)}\sqrt{\gamma}^{(n)}_{(k)}R^{(n)}_{(k)} - \alpha^{(n)}_{(k)}(\pi^{(n)}_{(k)})^2/(2\sqrt{\gamma}^{(n)}_{(k)}) + \alpha^{(n)}_{(k)}\pi^{ab(n)}_{(k)}\pi_{ab}{}^{(n)}_{(k)}/\sqrt{\gamma}^{(n)}_{(k)}$$
$$- 2\beta_{a}{}^{(n)}_{(k)}(\widehat{\delta}^{\langle 1\rangle}_{b}\pi^{ab(n)}_{(k)}) - 2\beta^{c(n)}_{(k)}\pi^{ab(n)}_{(k)}\Gamma^{\langle 1\rangle}{}_{cab}{}^{(n)}_{(k)}, \tag{1}$$

then the discretized ADM formulation is calculated as

$$\mathcal{H}^{\mathrm{ADM}(n)}_{(k)} \equiv \sqrt{\gamma}^{(n)}_{(k)}R^{(n)}_{(k)} + (\pi^{(n)}_{(k)})^2/(2\sqrt{\gamma}^{(n)}_{(k)}) - \pi^{ij(n)}_{(k)}\pi_{ij}{}^{(n)}_{(k)}/\sqrt{\gamma}^{(n)}_{(k)}, \tag{2}$$

$$\mathcal{M}^{\mathrm{ADM}(n)}_{\ell\,(k)} \equiv -2\gamma_{\ell i}{}^{(n)}_{(k)}(\widehat{\delta}^{\langle 1\rangle}_{j}\pi^{ij(n)}_{(k)}) - 2\pi^{ij(n)}_{(k)}\Gamma^{\langle 1\rangle}{}_{\ell ij}{}^{(n)}_{(k)}, \tag{3}$$

$$\frac{\gamma_{ij}{}^{(n+1)}_{(k)} - \gamma_{ij}{}^{(n)}_{(k)}}{\Delta t} = \cdots, \tag{4}$$

$$\frac{\pi^{ij(n+1)}_{(k)} - \pi^{ij(n)}_{(k)}}{\Delta t} = \cdots. \tag{5}$$

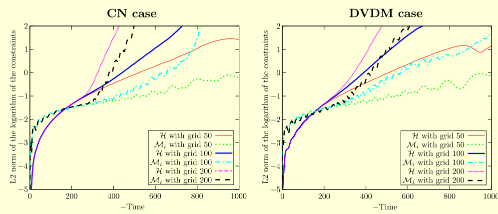
Numerical Tests

Following the proposal of the Apples-with-Apples, We show damping of constraint in numerical evolutions using polarized Gowdy wave evolution, which is one of the standard tests for comparisons of formulations in numerical relativity as is known to the Apples-with-Apples testbeds (Class. Quantum Grav. 21 (2004) 589).
The metric of polarized Gowdy wave is

$$ds^2 = t^{-1/2}e^{\lambda/2}(-dt^2 + dx^2) + t(e^{P}dy^2 + e^{-P}dz^2), \tag{6}$$

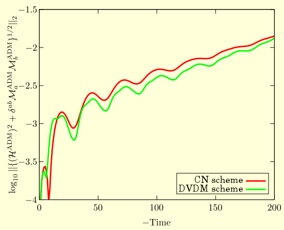
where P and λ are functions of x and t. The time coordinate t is chosen such that time increases as the universe expands, this metric is singular at $t = 0$ which corresponds to the cosmological singularity.

- Convergences



Left panel is using CN scheme, Right panel is using DVDM scheme. These values of the cases of 50 plot and 200 plot are rescaled by 1/4 and 4 times, respectively. Both of the convergences are satisfied until $t = -200$.

- Comparison of CN scheme with DVDM scheme



The violations of the constraints, $\{(\mathcal{H}^{\mathrm{ADM}})^2 + \delta^{ab}\mathcal{M}^{\mathrm{ADM}}_{a}\mathcal{M}^{\mathrm{ADM}}_{b}\}^{1/2}$, of the DVDM scheme (blue line) is lower than that of the CN scheme (red line).

Summary

- We proposed a discretized ADM formulation using the DVDM scheme.
- We performed some simulations using the DVDM scheme and CN scheme, and the violations of the DVDM scheme are lower than that of the CN scheme.